Skewing formulas for Delta Conjecture

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Outline

Choose two numbers $k \leq n$, define $K = k(n-k+1)$. In this talk, I will explain various ingredients in the following table:

	Degree	Algebra	Combinatorics	Geometry	Module
(1)	K	$E_{K,k} \cdot 1$	$\mathrm{PF}_{K,k}$	$Y_{n,k}$	$H_*^{BM}(Y_{n,k}) \circlearrowleft S_K$
(2)	n	$\Delta'_{e_{k-1}} e_n$	$\mathcal{LD}_{n,k}^{\mathrm{stack}}$	$X_{n,k}$	$H_*^{BM}(X_{n,k}) \circlearrowleft S_n$

The row (1) presents a symmetric function of degree K which is related to **Compositional Rational Shuffle Conjecture**. This symmetric function also appears as a character of the S_K action in the Borel-Moore homology of some space $Y_{n,k}$.

Similarly, the row (2) presents a symmetric function of degree n which is related to **Delta Conjecture**, and the homology of another space $X_{n,k}$.

Outline

Theorem (Gillespie-G.-Griffin)

All symmetric functions in row (1) agree.
All symmetric functions in row (2) agree.

Theorem (Gillespie-G.-Griffin)

We have $s_\lambda^\perp(1) = (2)$ where $\lambda = (k-1)^{n-k}$ is the rectangular Young diagram, and $s_\lambda^\perp$ is adjoint to multiplication by s_λ with respect to the Hall inner product.

Note that the operator $s_\lambda^\perp$ decreases the degree by $(k-1)(n-k) = K - n$.

Shuffle Conjecture

The case $k = n$ of the table corresponds to the celebrated **Shuffle Conjecture** proposed by Haiman, Haglund, Loehr, Remmel and Ulyanov, and first proved by Carlsson and Mellit.
On the **algebraic side** we have the symmetric function ∇e_n where e_n is the elementary symmetric function and ∇ is the operator which diagonalizes in the basis of **modified Macdonald polynomials**:

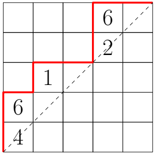
$$\nabla \widetilde{H}_\lambda = q^{n(\lambda)} t^{n(\lambda')} \widetilde{H}_\lambda.$$

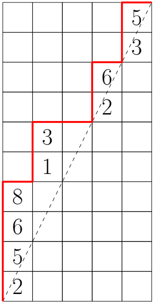
Here $q^{n(\lambda)} t^{n(\lambda')}$ is the product of (q,t)-contents of boxes in the diagram of λ. For example, $\nabla \widetilde{H}_{3,2} = q^4 t^2 \widetilde{H}_{3,2}$.

t	qt	
1	q	q^2

Shuffle Conjecture

On the combinatorial side we consider Dyck paths in the $n \times n$ square that stay weakly above the northeast diagonal in the grid. A **word parking function** is a labeling of the vertical runs of the Dyck path by positive integers such that the labeling strictly increases up each vertical run (but letters may repeat between columns; hence "word" parking function). We let $\mathrm{WPF}_{n,n}$ be the set of word parking functions.





Later we will consider word parking functions in arbitrary rectangles, see right picture.

Shuffle Conjecture

Theorem (Shuffle Conjecture, Carlsson-Mellit)

We have

$$\nabla e_n = \sum_{P \in \mathrm{WPF}_{n,n}} t^{\mathrm{area}(P)} q^{\mathrm{dinv}(P)} x^P,$$

where area *and* dinv *are certain statistics on word parking functions.*

Theorem (Hikita, G.-Mazin-Vazirani)

There is an algebraic variety X_n with an action of S_n in the (Borel-Moore) homology of X_n such that:
a) X_n has an affine paving with cells in bijection with parking functions. The dimension of the cell equals dinv *of the parking function.*
b) The Frobenius character of $H_*(X_n)$ equals ∇e_n.

Rational Shuffle Conjecture

We will need a generalization of the Shuffle Theorem known as the **Compositional Rational Shuffle Theorem**, conjectured by Bergeron-Garsia-Leven-Xin and proved by Mellit. On the combinatorial side we have the sum over rational parking functions in the $a \times b$ rectangle, while the algebraic side uses a certain operator $E_{a,b}$ from the **Elliptic Hall Algebra**.

Theorem (Mellit)

$$E_{a,b} \cdot 1 = \sum_{P \in \mathrm{WPF}_{a,b}} q^{\mathrm{area}(P)} t^{\mathrm{dinv}(P)} x^P.$$

Example

For $a = b = n$ one can check that $E_{a,b} \cdot 1 = \nabla e_n$, and we recover Shuffle Theorem.

Delta Conjecture

Next, we discuss **Delta Conjecture**, proposed by Haglund, Remmel and Wilson, and recently proved by Blasiak-Haiman-Morse-Pun-Seelinger, and D'Adderio-Mellit.

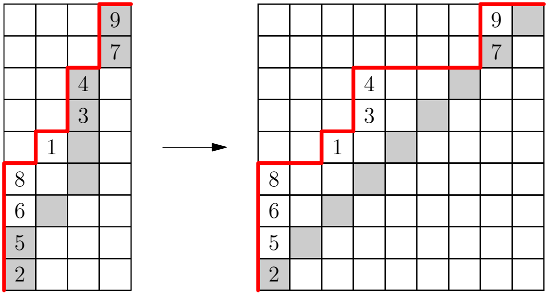
On the algebraic side we have the symmetric function $\Delta'_{e_{k-1}} e_n$ where $\Delta'_{e_{k-1}}$ is the operator on $\Lambda(q,t)$ which is diagonal in the modified Macdonald basis $\widetilde{H}_\lambda$ with eigenvalues

$$\Delta'_{e_{k-1}} \widetilde{H}_\lambda = e_{k-1}[B'_\lambda] \widetilde{H}_\lambda, \quad B'_\lambda = \sum_{\square \in \lambda,\, \square \neq (0,0)} q^{a'(\square)} t^{\ell'(\square)}.$$

Delta Conjecture

On the combinatorial side we have **stacked parking functions**. A **stack** S of boxes in an $n \times k$ grid is a subset of the grid boxes such that there is one element of S in each row, at least one in each column, and each box in S is weakly to the right of the one below it.

A (word) **stacked parking function** with respect to S is a labeled up-right path D such that each box of S lies below D, and the labeling is strictly increasing up each column.



Delta Conjecture

Theorem (Delta Conjecture)

$$\Delta'_{e_{k-1}} e_n = \sum_{P \in \mathcal{LD}^{\text{stack}}_{n,k}} q^{\text{area}(P)} t^{\text{hdinv}(P)} x^P .$$

Theorem (Gillespie-G.-Griffin)

Letting $K = k(n-k+1)$ *and* $\lambda = (k-1)^{n-k}$*, we have*

$$\Delta'_{e_{k-1}} e_n = s_\lambda^\perp (E_{K,k} \cdot 1), \qquad (1)$$

where $s_\lambda^\perp$ *is the adjoint to multiplication by the Schur function* s_λ*.*

We give two proofs of this theorem, by applying $s_\lambda^\perp$ both to the combinatorial and algebraic sides of the Compositional Rational Shuffle Theorem for (K, k). As a consequence, we obtain a new proof of Delta Conjecture.

Algebraic proof

The algebraic proof relies on results of Blasiak, Haiman, Morse, Pun and Seelinger [BHMPS]. Let f be a (Laurent) polynomial in k variables, define

$$\sigma(f) = \sum_{w \in S_k} w\left(\frac{f}{\prod_{i<j}(1 - x_j/x_i)}\right),$$

and

$$H_{q,t}^k(f) = \sigma\left(\frac{f\prod_{i<j}(1 - qtx_i/x_j)}{\prod_{i<j}(1 - qx_i/x_j)(1 - tx_i/x_j)}\right).$$

Algebraic proof

Let $\pi_k : \Lambda \to \Lambda_k$ denote the restriction of symmetric functions to k variables, and pol denotes the polynomial part of a symmetric Laurent series. The following can be deduced from the results of BHMPS.

Theorem (BHMPS+ε)

One has

a)

$$\pi_k(\omega E_{K,k}(1)) = H^k_{q,t}\left(\frac{x_1^{n-k+1}\cdots x_k^{n-k+1}}{\prod(1-qtx_i/x_{i+1})}\right)_{\mathrm{pol}}.$$

b)

$$\pi_k(\omega\Delta'_{e_{k-1}}e_n) = H^k_{q,t}\left(\frac{x_1\cdots x_k h_{n-k}(x_1,\ldots,x_k)}{\prod(1-qtx_i/x_{i+1})}\right)_{\mathrm{pol}}.$$

c) The Schur functions appearing in the expansion of left hand sides of (a) and (b) have at most k parts, so applying π_k does not lose any information.

Algebraic proof

The algebraic proof essentially follows from this theorem and the following observation:

Lemma

We have

$$s_{(n-k)^{k-1}}(x_1, \ldots, x_k) = \sum_{\substack{\mu_i \le n-k \\ |\mu| = (n-k)(k-1)}} x_1^{\mu_1} \cdots x_k^{\mu_k}.$$

and

$$s_{(n-k)^{k-1}}(x_1^{-1}, \ldots, x_k^{-1}) = \frac{h_{n-k}(x_1, \ldots, x_k)}{x_1^{n-k} \cdots x_k^{n-k}}.$$

Combinatorial proof: idea

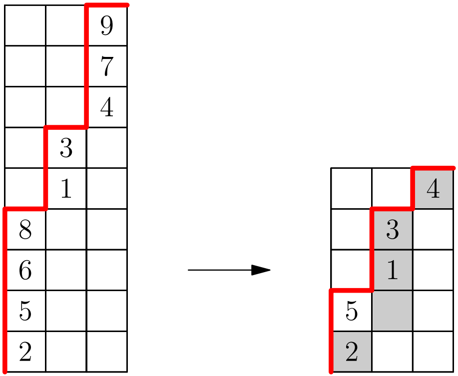
Before getting into the details of the combinatorial proof, we need to find a way to get from (K, k) parking functions to stacked parking functions in the $k \times n$ rectangle. Given a (K, k) (word) parking function, we split the labels in each vertical run into "big" and "small" such that the b_i big labels are above s_i small labels, and there are $\sum b_i = K - n$ big ones. We say that a configuration of big and small labels is **admissible** if $b_i \leq n - k$ for all i.

- The parking function $F(\pi)$ is obtained by erasing all big labels in π and deleting all vertical steps of the Dyck path to the left of these labels.
- The heights of the stacks are given by $w_i = n - k + 1 - b_i$.

Lemma

The result $F(\pi)$ is a valid stacked parking funcition.

Example



Here $n = 5, k = 3$, so $K = k(n - k + 1) = 9$. We split the labels into "big" ($> n$) and "small" ($\leq n$). Erase the big labels, the result is a labeled path in $k \times n$ rectangle. The stack remembers how many boxes we erased in each column.

Combinatorial proof

We fix a $K \times k$ Dyck path D and denote

$$f_D = \sum_{\pi \in \mathrm{WPF}_{K,k}(D)} q^{\mathrm{tdinv}(\pi)} x^{\pi}.$$

where the sum is over word parking functions. It is easy to see that

$$f_D[X+Y;q] = \sum_{\underline{b},\underline{s}} q^{d(\underline{s},\underline{b})} f_{D,\underline{s}}[X;q] f_{D,\underline{b}}[Y;q]$$

where the sum is over all possible decompositions of vertical steps of D into big and small. Here $f_{D,\underline{b}}, f_{D,\underline{s}}$ are constributions of "big" and "small" labels, and $d(\underline{s},\underline{b})$ counts inversion pairs between big and small. Now

$$s^{\perp}_{(k-1)^{n-k}} f_D = \langle s_{(k-1)^{n-k}}(Y), f_D[X+Y;q] \rangle =$$

$$\sum_{\underline{b},\underline{s}} q^{d(\underline{s},\underline{b})} f_{D,\underline{s}}[X;q] \langle s_{(k-1)^{n-k}}(Y), f_{D,\underline{b}}[Y;q] \rangle.$$

Combinatorial proof

The following result is the technical core of the combinatorial proof:

Theorem (Gillespie-G.-Griffin)

Let D be a Dyck path and $\underline{b}$ be an admissible sequence (so $b_i \leq n-k$ and at most the number of vertical steps of D in column i). Then

$$\langle f_{D,\underline{b}}[X;q], s_{(k-1)^{n-k}} \rangle = q^{c_{D,\underline{b}}}.$$

If $\underline{b}$ is not admissible, then $\langle f_{D,\underline{b}}[X;q], s_{(k-1)^{n-k}} \rangle = 0$.

Combinatorial proof

We use Jacobi-Trudi formula to write $s_{(k-1)^{n-k}}$ as an alternating sum of $h_{\widetilde{\alpha}}$ over some set of compositions $\widetilde{\alpha}$ which we call **allowable contents**. The pairing $\langle h_{\widetilde{\alpha}}, f_{D,\underline{b}}[X;q]\rangle$ equals the coefficient of $f_{D,\underline{b}}[X;q]$ at the monomial symmetric function $m_{\widetilde{\alpha}}$, which counts the column-strict fillings of $(D,\underline{b})$ with content $\widetilde{\alpha}$.

We define a sign-reversing involution φ and prove the following:

- φ is an involution on the set of column-strict fillings with allowable contents
- φ has a unique fixed point, which we denote $P^0_{D,\underline{b}}$, that has positive sign,
- φ preserves the statistics $\mathrm{tdinv}_{\mathrm{big}}$ and reverses the sign $(-1)^{\mathrm{sgn}(\widetilde{\alpha})}$ for every element except $P^0_{D,\underline{b}}$.

To complete the proof, we also need to compute $\mathrm{tdinv}_{\mathrm{big}}(P^0_{D,\underline{b}})$ which turns out to be quite nontrivial.

Remarks

- For $t = 0$, the skewing formula was proved by Gillespie and Griffin in a previous paper (which was an inspiration for this project). Note that there is only one Dyck path of area zero, so (K, k) parking functions of area zero can be identifed with column-strict tableaux in the $k \times (K/k)$ rectangle.
- By the work of Lascoux, Leclerc and Thibon, the coefficient $\langle f_{D,\underline{b}}[X; q], s_{(k-1)^{n-k}} \rangle$ equals certain parabolic Kazhdan-Lusztig polynomial. In particular, its coefficients are nonnegative. It would be interesting to see why it is in fact equal to a very specific monomial in q.
- There is another version of the Delta Conjecture due to Haglund, Remmel, and Wilson, called the Valley Delta Conjecture, that remains open. Can the skewing formula be used to prove the Valley Delta Conjecture?
- Are there other skewing formulas related to combinatorics of Macdonald polynomials?

Thank You!

Geometry

We define a certain subvariety $Y_{n,k}$ in the affine flag variety for GL_K and prove the following:

Theorem (Gillespie-G.-Griffin)

a) The space $Y_{n,k}$ has an affine cell decomposition with cells in bijection with (K,k) rational parking functions.
b) The dimension of the cell is equal (up to a constant) to the dinv statistic of the parking function.
c) There is an S_K action in the Borel-Moore homology of $Y_{n,k}$, and the corresponding Frobenius character equals $E_{K,k} \cdot 1$.

Geometry

We define another subvariety $X_{n,k}$ in the partial affine flag variety and provve the following:

Theorem (Gillespie-G.-Griffin)

There is an S_n action in the Borel-Moore homology of $X_{n,k}$, and the corresponding Frobenius character equals $\Delta'_{e_{k-1}} e_n$.

In fact, we prove that $s_\lambda^\perp \mathrm{grFrob} H^*(Y_{n,k}) = \mathrm{grFrob} H^*(X_{n,k})$.